

The normal closure of a power of a half-twist has infinite index in the mapping class group of a punctured sphere

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Abstract

In this paper we show that the normal closure of the m^{th} power of a half-twist has infinite index in the mapping class group of a punctured sphere. Furthermore, in some cases we prove that the quotient of the mapping class group of the punctured sphere by the normal closure of a power of a half-twist contains a free abelian subgroup. As a corollary we prove that the quotient of the hyperelliptic mapping class group of a surface of genus at least two by the normal closure of the m^{th} power of a Dehn twist has infinite order, and for some integers m the quotient contains a free nonabelian subgroup. As a second corollary we recover a result of Coxeter: the normal closure of the m^{th} power of a half-twist in the braid group of at least five strands has infinite index if n is at least four. Our method is to reformulate the Jones representation of the mapping class group of a punctured sphere, using the action of Hecke algebras on W -graphs, as introduced by Kazhdan-Lusztig.

1 Introduction

Let $\Sigma_{g,n}$ be a surface of genus g with n marked points. The mapping class group of $\Sigma_{g,n}$, denoted by $\text{Mod}(\Sigma_{g,n})$, consists of those isotopy classes of homeomorphisms that preserve both the orientation of $\Sigma_{g,n}$, and the set of marked points. When $n = 0$, we will simply write Σ_g . If $g = 0$ the mapping class group $\text{Mod}(\Sigma_{0,n})$ is generated by half-twists (homeomorphisms that interchange two marked points of $\Sigma_{0,n}$). The main result of this paper is the following.

Theorem A *The normal closure of the m^{th} power of a half-twist has infinite index in $\text{Mod}(\Sigma_{0,n})$ if $n \geq 6$ is even and $m \geq 5$.*

The proof of Theorem A relies on the Jones representation of the mapping class group $\text{Mod}(\Sigma_{0,n})$, where $n \geq 6$ is even [18]. The following theorem was suggested by Funar.

Theorem B *The quotient $\text{Mod}(\Sigma_{0,n})$ by the closure of the m^{th} power of a half-twist contains a free nonabelian subgroup, if $n \geq 6$ is even, and $m \notin \{2, 4, 6, 10\}$ if m is even, and $m \notin \{1, 3, 5\}$ if m is odd.*

The proof of Theorem B we give in this paper uses the Jones representation of the mapping class group of the sphere with marked points. As pointed out by Funar, there is an alternate proof of Theorem B in the case where m even. Funar-Kohno used quantum representations to prove that the quotient of the mapping class group of the sphere with 4 boundary components by the normal closure of the k^{th} power of Dehn twists contains

free nonabelian subgroup [12, Proposition 3.2]. Therefore, the quotient of the mapping class group of the sphere with n boundary components by the normal closure of the k^{th} power of Dehn twists contains free nonabelian subgroup. The surjection of the latter quotient on the quotient $\text{Mod}(\Sigma_{0,n})$ by the closure of the $2k^{th}$ power of a half-twist has finite kernel generated by the Dehn twists about the boundary components; the image contains a free nonabelian subgroup.

Results for the hyperelliptic mapping class group. Let c be a nonseparating simple closed curve in Σ_g ; denote by T_c a Dehn twist about c . In Birman's problem list, she asked whether the normal closure of T_c^2 in $\text{Mod}(\Sigma_g)$ has infinite index if $g \geq 3$ [3, Question 28]. It is well known that the normal closure of T_c^2 has finite index in $\text{Mod}(\Sigma_g)$ when $g = 1$ or 2 .

Humphries answered Birman's question by proving that, in fact, the normal closure of T_c^2 has finite index in $\text{Mod}(\Sigma_g)$ [15, Theorem 1]. Let $\text{SMod}(\Sigma_g)$ denote the hyperelliptic mapping class group, that is, those elements of $\text{Mod}(\Sigma_g)$ that commute with a fixed hyperelliptic involution (an element of order 2 of $\text{Mod}(\Sigma_g)$ that acts as $-\text{Id}$ on $H_1(\Sigma_g)$). In the same paper Humphries used the fact that $\text{Mod}(\Sigma_2) = \text{SMod}(\Sigma_2)$ to show that if $m \geq 4$, then the normal closure of T_c^m has infinite index in $\text{Mod}(\Sigma_2)$ [15, Theorem 4]. As a corollary of the Theorem A, we extend Humphries' result for $\text{SMod}(\Sigma_g)$ to the case $g > 2$.

Corollary 1.1. *Let $T_c \in \text{SMod}(\Sigma_g)$ be a Dehn twist about a nonseparating simple closed curve c . If $m \geq 5$, and $g \geq 2$ then the normal closure of T_c^m has infinite index in $\text{SMod}(\Sigma_g)$.*

We note that the curve c in the theorem above is symmetric, that is, it remains fixed under the action of a fixed hyperelliptic involution.

Proof. By Birman-Hilden's theorem we have that the map $\text{SMod}(\Sigma_g) \rightarrow \text{Mod}(\Sigma_{0,2g+2})$ is a surjective homomorphism [4, Theorem 1]. The result follows from the fact that the normal closure of T_c^n in $\text{SMod}(\Sigma_g)$ surjects onto the normal closure of a half-twist in $\text{Mod}(\Sigma_{0,n})$. \square

Our methods in the proof of Theorem A are inspired by Humphries' original proof, which relies on Jones' explicit calculation of his representation of $\text{Mod}(\Sigma_{0,2g+2})$ when $g = 2$. Unfortunately, the explicit computation of the matrices when $g > 2$ is difficult. We avoid this difficulty by finding a certain block form of the representation in the general case.

We note that it is not obvious how to extend the Jones representation from $\text{SMod}(\Sigma_g)$ to the whole mapping class group $\text{Mod}(\Sigma_g)$ (see for example [19, Section 2.3]), and hence Humphries' techniques cannot be used for $\text{Mod}(\Sigma_g)$ when $g > 2$. However, Funar successfully used TQFT representations of the mapping class group to prove that the normal closure of T_c^m has infinite index in $\text{Mod}(\Sigma_g)$, if $g \geq 2$ and if $m \neq 2, 3, 4, 6, 8, 12$ [11, Corollary 1.2].

Funar-Kohno improved Funar's result above by proving that if $g \geq 3$, and $m \notin \{3, 4, 8, 12, 16, 24\}$ or $g = 2$, m even and $m \notin \{4, 8, 12, 16, 24, 40\}$ the quotient of $\text{Mod}(\Sigma_g)$ by the normal closure of the m^{th} power of a Dehn twist contains a free nonabelian subgroup [12]. We can use Theorem B to prove a similar theorem for the hyperelliptic mapping class group.

Corollary 1.2. *We assume that $g \geq 2$, $m \geq 4$, and $m \notin \{1, 2, 3, 4, 6, 10\}$ is even. Then the quotient of $\text{SMod}(\Sigma_g)$ by the normal closure of the m^{th} power of a Dehn twist contains a free nonabelian subgroup.*

The proof of Corollary 1.2 is the same as the proof of Corollary 1.1.

Remarks on Theorems A and B. Theorem B gives a stronger result than Theorem A in the case where the power m is even. The methods we use to prove them are different. The proof of Theorem A uses an explicit matrix calculation that we develop in this paper. In fact we give a new approach to the Jones representation by using a different definition of the Hecke algebras and we use the notion of W-graphs introduced by Kazhdan-Lusztig. In the proof of Theorem B we show how to modify the Burau representation so that its image is contained in the Jones representation.

Quantum representations deal with the quotient of the mapping class group of the sphere with boundary components by the normal closure of powers of Dehn twists. The representations we construct in this paper deal with representations of the quotient of the mapping class group $\text{Mod}(\Sigma_{0,n})$ of the sphere with even number of marked points by the normal closure of powers of half-twists. One could find relations between the two representations, and this may give new representations of the quotient of $\text{Mod}(\Sigma_{0,n})$ by the normal closure of powers of half-twists. These representation might give more information about the latter quotient.

Results for braid groups. We denote the disc with n marked points in its interior by D_n . The mapping class group $\text{Mod}(D_n)$ of D_n consists of isotopy classes of homeomorphisms that preserve both the orientation of D_n , the set of marked points, and fix the boundary pointwise. The braid group B_n is isomorphic to $\text{Mod}(D_n)$. It is a well known result due to Artin that B_n is generated by half-twists, that is, homeomorphisms that interchange two marked points [2, Sections 1.2, 1.3].

A similar result as in Theorem A holds for the braid group B_n . Coxeter used hyperbolic geometry to prove that the normal closure of the m^{th} power of a half-twist has finite index in B_n if and only if $(n-2)(m-2) < 4$ [6, Section 10]. As a second corollary of our Theorem A, we recover Coxeter's theorem when $n \geq 4$ and $m \geq 4$.

Corollary 1.3. *The normal closure of the m^{th} power of a half-twist has infinite index in the braid group B_n , if $m \geq 5$, and $n \geq 4$.*

The proof of Corollary 1.3 is independent from Coxeter's proof.

Construction of the Jones representation. Let $H(q, 2g+2)$ be a Hecke algebra with complex parameter q . There is a representation $B_{2g+2} \rightarrow H(q, 2g+2)$ from the braid group into the group of units of $H(q, 2g+2)$. We can think of $H(q, 2g+2)$ as a quotient of the group algebra of B_{2g+2} over $\mathbb{Z}[q^{\pm 1}]$. Thus, any representation of $H(q, 2g+2)$ will give a representation for B_{2g+2} . We can think of $\text{Mod}(\Sigma_{0,2g+2})$ as a quotient group of the braid group B_{2g+2} . Jones observed that in some cases we can modify the representations of $H(q, 2g+2)$ so that we can define representations for $\text{Mod}(\Sigma_{0,2g+2})$ [18, Section 10].

Assume that q is not a root of unity. The set of irreducible representations of the Hecke algebra $H(q, 2g+2)$ is in bijective correspondence with the set of Young diagrams of size $2g+2$. When the Young diagram has the shape of a rectangle, we will show that under a modification, the corresponding irreducible representation of $H(q, 2g+2)$ gives a representation of $\text{Mod}(\Sigma_{0,2g+2})$. We will also explain a method for explicitly computing matrices of the irreducible representations of $H(q, 2g+2)$ in this case by using the notion of W-graphs (see Section 3.2). If $g = 2$ we explicitly calculate the matrices of the representation of $\text{Mod}(\Sigma_{0,2g+2})$. Our calculations are equivalent to those of Jones, but we make different choices of parameters and hence the resulting matrices are slightly

different. When $g \geq 3$, the calculations are much more complicated and we will not compute the full matrices explicitly. However, we will show that the matrices have a particular block form for $g \geq 3$ that is sufficient for the calculations we require (see Theorem 3.6).

Outline of the paper. In Section 2 we give basic background for braid groups, hyperelliptic mapping class groups, and Hecke algebras. In Section 3 we define representations of Hecke algebras, and consequently of braid groups. More particularly, in Section 3.1 we give relations between irreducible representations of Hecke algebras, and Young diagrams. Sections 3.2 and 3.3 are devoted to the construction of $\mathbb{Z}[q^{\pm 1}]$ -modules, and an action of Hecke algebras on these modules by using the notion of W-graphs; as a result we end up with well-defined irreducible representations of Hecke algebras. In Section 3.3 we explain how Young diagrams are related to these $\mathbb{Z}[q^{\pm 1}]$ -modules. In Section 3.4 we examine certain properties of representations of B_n we obtain via the homomorphism $B_n \rightarrow H(q, n)$. In Section 4 we define the representation of $\text{SMod}(\Sigma_{0,2g+2})$. In Section 5 we prove Theorem A, and Corollary 1.3. In Section 6 we prove Theorem B.

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2 Preliminaries

In this section we give basic facts about braid groups, Hecke algebras and the hyperelliptic mapping class group.

The braid group and mapping class group of the punctured sphere. Consider a disc D_n with n distinct marked points in its interior as indicated in Figure 1. The braid group B_n is defined to be $\text{Mod}(D_n)$. We denote by p_i the marked points of D_n , enumerating from left to right. For $i \leq n-1$ we denote by σ_i the mapping classes of D_n obtained by interchanging the points p_i and p_{i+1} by rotating clockwise as depicted in Figure 1. The

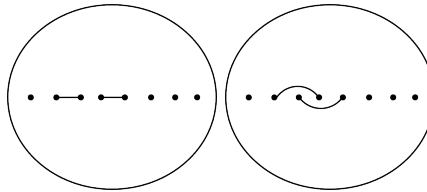


Figure 1: The action of the half-twists σ_3 on two arcs connecting points p_2, p_3 and p_4, p_5 respectively.

elements σ_i are called *half-twists*, and they generate B_n . More precisely, we have the following presentation for B_n :

$$\langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, [\sigma_i, \sigma_j] = 1 \text{ if } |i - j| > 1 \rangle.$$

Consider the natural epimorphism $B_n \rightarrow S_n$, where S_n is the symmetric group. More precisely, we have $B_n / \langle \sigma_i^2 \rangle \cong S_n$. We denote by s_i the transpositions $(i, i + 1)$ in S_n . We have the following presentation for S_n :

We denote by S the set $\{s_i \mid 1 \leq i \leq n-1\}$; the pair (S_n, S) forms a *Coxeter system* for the symmetric group (see [5] for general definition).

Consider the sphere with n punctures $\Sigma_{0,n}$ induced from D_n by gluing a disc to the boundary of D_n . By extending every homeomorphism of D_n as the identity in $\Sigma_{0,n}$ we obtain a surjective homomorphism

Every half-twist in B_n is mapped into a half-twist in $\text{Mod}(\Sigma_{0,n})$. We will denote the generators of $\text{Mod}(\Sigma_{0,n})$ by H_i . Thusly, the homomorphism above is defined by $\sigma_i \mapsto H_i$.

The group $\text{Mod}(\Sigma_{0,n})$ has a presentation with generators H_1, H_2, \dots, H_{n-1} , and relations

and

where $1 \leq i \leq n-2$ and $1 \leq k, j \leq n-1$ [9, Section 5.1.3]

Hyperelliptic mapping class group. The mapping class group $\text{Mod}(\Sigma_g)$ is generated by Dehn twists about the curves $c_0, c_1, c_2, \dots, c_{2g}$ depicted in Figure 2 [17].

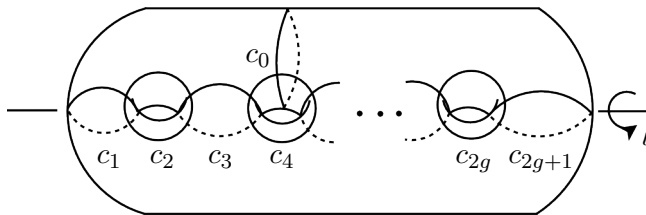


Figure 2: Generators for $\text{Mod}(\Sigma_q)$.

We think of Σ_g as a branched cover of the sphere \mathbb{S}^2 , over $2g + 2$ points as follows: let ι be the hyperelliptic involution of the surface Σ_g as indicated in Figure 2. Obviously $\iota \in \text{Mod}(\Sigma_g)$. Then the quotient Σ_g/ι is an orbifold sphere with $2g + 2$ cone points of order 2. In this paper we will not need this geometric information about Σ_g/ι , and we will consider it as a topological space $\Sigma_{0,2g+2}$, that is, a sphere with $2g + 2$ marked points.

We denote by $\text{SMod}(\Sigma_g)$ the subgroup of $\text{Mod}(\Sigma_g)$ consisting of mapping classes that commute with ι . Birman-Hilden proved the following exact sequence [4, Theorem 1].

$$1 \rightarrow \langle \iota \rangle \rightarrow \mathrm{SMod}(\Sigma_q) \rightarrow \mathrm{Mod}(\Sigma_{0,2q+2}) \rightarrow 1$$

The group $\text{SMod}(\Sigma_g)$ is called *hyperelliptic mapping class group*, and it is generated by $T_{c_1}, T_{c_2}, \dots, T_{c_{2g+1}}$. The map $\text{SMod}(\Sigma_g) \rightarrow \text{Mod}(\Sigma_{0,2g+2})$ is defined by $T_{c_i} \mapsto H_i$.

For $0 \leq i \leq 2g+1$ consider the curves c_i indicated in Figure 2. Then it is obvious that if $g = 2$, then $\text{SMod}(\Sigma_2) = \text{Mod}(\Sigma_2)$. Also, since $T_{c_0}^m$ is not an element of $\text{SMod}(\Sigma_g)$ for any $m \geq 1$, then $\text{SMod}(\Sigma_g)$ has infinite index in $\text{Mod}(\Sigma_g)$.

Hecke Algebras. Let $\mathbb{Z}[q^{\pm 1}]$ be the ring of Laurent polynomials over \mathbb{Z} where q is an indeterminate. Consider the Coxeter system for the symmetric group (S_n, S) described above, where S consists of the transpositions $s_i = (i, i+1)$. We denote by $H(q, n)$ the algebra over $\mathbb{Z}[q^{\pm 1}]$ generated by the set $\{T_{s_i} \mid s_i \in S\}$ with relations as follows:

- (1) We have $T_{s_i} T_{s_{i+1}} T_{s_i} = T_{s_{i+1}} T_{s_i} T_{s_{i+1}}$.
- (2) If $|i - j| > 1$, then $T_{s_i} T_{s_j} = T_{s_j} T_{s_i}$.
- (3) For all $s_i \in S$, we have $T_{s_i}^2 = 1 + (q - q^{-1})T_{s_i}$.

An algebra $H(q, n)$ of this form is called a *Hecke algebra*. Let (S_n, S) be a Coxeter system. Given $w \in S_n$ we can write $w = s_{j_1} s_{j_2} \dots s_{j_p}$, where $s_{j_k} \in S$. If p is minimal, we say that this is a *reduced expression* for w ; then $l(w) = p$ is called the length of w . If $l(w) = p$ then we define $T_w = T_{s_{j_1}} T_{s_{j_2}} \dots T_{s_{j_p}}$. The element T_w is independent of the choice of the reduced expression for w . It turns out that the multiplication rule is the following, [13, Lemma 4.4.3]:

$$T_{s_i} T_w = \begin{cases} T_{s_i w}, & \text{if } l(s_i w) > l(w), \\ T_{s_i w} + (q - q^{-1})T_w, & \text{if } l(s_i w) < l(w). \end{cases}$$

Furthermore, $H(q, n)$ admits a basis $\{T_w \mid w \in S_n\}$ [13, Theorem 4.4.6].

We can easily check that $T_{s_i}^{-1} = T_{s_i} - (q - q^{-1})$. The map $\psi : B_n \rightarrow H(q, n)$ defined by $\psi(\sigma_i) = T_{s_i}$ is a well defined homomorphism from B_n to the group of units of $H(q, n)$. Similarly if $q = 1$ then we have a well defined homomorphism $\phi : S_n \rightarrow H(1, n)$ defined by $\phi(s_i) = T_{s_i}$. We can think of $H(q, n)$ as a quotient of group algebra of B_n , and $H(1, n)$ as the group algebra of S_n .

3 Braid group representations

Our aim in this section is to define representations of $H(q, n)$. Since $B_n \rightarrow H(q, n)$ is a well defined homomorphism, any linear representation of $H(q, n)$ will give a rise to a representation of B_n . It is known that the set of irreducible representations of $H(q, n)$ is in bijective correspondence with the set of irreducible representations of S_n [13, Theorem 8.1.7]. But irreducible representations of S_n are in bijective correspondence with Young diagrams [7, Theorem 1]. The construction of the representations of $H(q, n)$ we are going to provide arises from the notion of W-graphs introduced by Kazhdan-Lusztig [20]. This representation is well defined for any q , even for roots of unity. For $i < n$ the generators of S_n and B_n will be denoted by s_i and σ_i respectively. As mentioned in Section 2, the generators s_i of S_n correspond to the transpositions $(i, i+1)$.

3.1 Young diagrams

A *Young diagram* $\lambda = [\mu_1, \mu_2, \dots, \mu_k]$ is an array of n boxes with μ_i boxes in the i^{th} row, $\mu_i \geq \mu_{i+1}$, and $\sum \mu_i = n$. We denote by Λ_n the set of all Young diagrams with n boxes,

of Figure 5, where in each box we have assigned its hook length. The cardinality of Y_λ is equal to

$$|Y_\lambda| = \frac{n!}{\prod_{x \in \lambda} \text{hook}(x)}.$$

For the diagram λ of Figure 5 we have $|Y_\lambda| = 68640$.

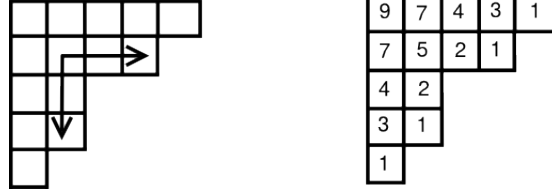


Figure 5: The hook length of every box in the Young diagram.

3.2 The Burau representation

For $n > 2$ the reduced Burau representation $\beta_t : B_n \rightarrow \text{GL}_{n-1}(\mathbb{Z}[t^{\pm 1}])$ is determined by the matrices

$$\beta_t(\sigma_1) = \begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix} \oplus I_{n-3}, \quad \beta_t(\sigma_{n-1}) = I_{n-3} \oplus \begin{pmatrix} 1 & 0 \\ t & -t \end{pmatrix}$$

and for $1 < i < n-1$

$$\beta_t(\sigma_i) = I_{i-2} \oplus \begin{pmatrix} 1 & 0 & 0 \\ t & -t & 1 \\ 0 & 0 & 1 \end{pmatrix} \oplus I_{n-i-2}.$$

A direct calculation shows that $(-\beta_t(\sigma_i))^2 = (t-1)(-\beta_t(\sigma_i)) + t$. This relation was first observed by Jones. For $q^2 = t$, the matrices $(-q^{-1}\beta_{q^2}(\sigma_i))$ satisfy the quadratic relation of $H(q, n)$. Since the Burau representation is irreducible and satisfies the quadratic relation, one can check that $(-q^{-1}\beta_{q^2}(\sigma_i)) = \pi_\lambda(\sigma_i)$, where λ is the Young diagram

3.3 W-graphs

In this section we provide a tool to explicit construct matrices of representations of the Hecke algebra $H(q, n)$. In fact, we will define a $\mathbb{Z}[q^{\pm 1}]$ -module E , and an action of $H(q, n)$ on E . A W-graph encodes the structure of a $\mathbb{Z}[q^{\pm 1}]$ -module in the sense that its vertices correspond to a basis for E ; the edges provide all the information we need for the action of $H(q, n)$ on E . First we will give a new basis for $H(q, n)$; as before each basis element is associated to an element of the symmetric group S_n . Then we will define an equivalence relation on elements of S_n . The equivalence classes will be called cells. Vertices of a W-graph correspond to elements of a fixed cell. We will define an action of generators of $H(q, n)$ on the basis of E . This action extends to a representation $H(q, n) \rightarrow \text{End}(E)$ [20].

Definition of W-graphs. Let (S_n, S) be the Coxeter system for the symmetric group introduced in the previous section. A *W-graph* is defined to be a set of vertices X and a set of edges Y together with the following data. For each vertex $x \in X$, we are given a subset $I_x \subset S$, and for each ordered pair of vertices (y, w) with $\{y, w\} \in Y$, we are given an integer $\mu(y, w)$, subject to the requirements in the following paragraph.

Let E be the free $\mathbb{Z}[q^{\pm 1}]$ -module with basis associated to the vertex set X . Recall that s_i is the transposition $(i, i+1)$ in S_n . For any $s_i \in S$, for any $w \in X$ (considering X as a basis for E) we define the map τ_{s_i} as follows:

$$\tau_{s_i} w = \begin{cases} -q^{-1}w, & \text{if } s_i \in I_w, \\ qw + \sum \mu(y, w)y, & \text{if } s_i \notin I_w, \end{cases}$$

where the sum is taken over all $y \in X, s_i \in I_y$ such that $\{y, w\} \in Y$. Extending linearly we get an endomorphism of E . For $i < n-2$, we require that

$$\tau_{s_i} \tau_{s_{i+1}} \tau_{s_i} = \tau_{s_{i+1}} \tau_{s_i} \tau_{s_{i+1}}$$

and for $|i-j| > 1$

$$\tau_{s_i} \tau_{s_j} = \tau_{s_j} \tau_{s_i}.$$

In other words, we require that the endomorphisms τ_{s_i} satisfy the usual relations in the braid group.

We note that Kazhdan-Lusztig defined W-graphs for any Coxeter system.

Example 1. We give an example of a W-graph for the group S_3 in Figure 6. We label the vertices of the W-graph by the elements of the set $X = \{s_1, s_2 s_1\}$. We let $I_{s_1} = \{s_1\}$, $I_{s_2 s_1} = \{s_2\}$ and the integers $\mu(s_1, s_2 s_1) = 1$, $\mu(s_2 s_1, s_1) = 0$. Furthermore, we have

$$\begin{aligned} \tau_{s_1}(s_1) &= -q^{-1}s_1 \\ \tau_{s_1}(s_2 s_1) &= qs_2 s_1 + s_1 \\ \tau_{s_2}(s_1) &= qs_1 + s_2 s_1 \\ \tau_{s_2}(s_2 s_1) &= -q^{-1}s_2 s_1 \end{aligned}$$

The matrices of the representation are as follows:

$$T_{s_1} \mapsto \begin{pmatrix} -q^{-1} & 1 \\ 0 & q \end{pmatrix}, T_{s_2} \mapsto \begin{pmatrix} q & 0 \\ 1 & -q^{-1} \end{pmatrix}.$$

We note that in the literature the vertices of W-graphs are often labeled by I_w where $w \in S_n$.

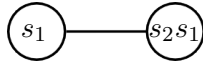


Figure 6: Example of a W-graph

The endomorphism τ_{s_i} satisfies the quadratic relation $(\tau_{s_i} + q^{-1})(\tau_{s_i} - q) = 0$. To see this consider $w \in S_n$ such that $s_i \in I_w$. Then we have:

$$(\tau_{s_i} + q^{-1})(\tau_{s_i} - q)(w) = (\tau_{s_i} + q^{-1})(-q^{-1}w - qw) = q^{-2}w + w - q^{-2}w - w = 0.$$

On the other hand, if $s_i \notin I_w$, then we have:

$$\begin{aligned}
(\tau_{s_i} + q^{-1})(\tau_{s_i} - q)(w) &= (\tau_{s_i} + q^{-1})(qw + \sum \mu(y, w)y - qw) \\
&= \sum \mu(y, w)\tau_{s_i}(y) + q^{-1} \sum \mu(y, w)y \\
&= -q^{-1} \sum (y, w)y + q^{-1} \sum \mu(y, w)y \\
&= 0
\end{aligned}$$

By the second condition of the definition of W -graphs, and since the endomorphism τ_{s_i} satisfies the quadratic relation, the map $T_{s_i} \mapsto \tau_{s_i}$ extends to a representation of the Hecke algebra $H(q, n)$.

The Kazhdan-Lusztig basis. Let $a \rightarrow \bar{a}$ be the involution on the ring $\mathbb{Z}[q^{\pm 1}]$ defined by $\bar{q} = q^{-1}$. We extend this to an involution on $H(q, n)$ by the formula

$$\overline{\sum a_w T_w} = \sum \bar{a} T_{w^{-1}}^{-1}.$$

We consider the set of generators S of S_n , and the length function $l : S_n \rightarrow \mathbb{Z}$ described in Section 2. We denote by \leq the Bruhat order on the set of all words in the generating set S , that is, $y \leq x$ if y is subword of x .

We have the following theorem of Kazhdan-Lusztig [20, Theorem 1] reformulated by Yin [21, Theorem 1.4]:

Theorem 3.1. *For any $w \in S_n$, there is a unique element $C_w \in H(q, n)$ such that $\overline{C_w} = C_w$, where*

$$C_w = \sum_{y \leq w} (-q)^{l(w)-l(y)} \overline{P_{y,w}(q^2)} T_y$$

and $P_{y,w}$ is a polynomial of degree at most $\frac{1}{2}(l(w) - l(y) - 1)$ if $y < w$, and $P_{w,w} = 1$.

The polynomial $P_{y,w}$ is known in the literature as the *Kazhdan-Lusztig polynomial*, and Theorem 3.1 proves the existence and uniqueness of $P_{y,w}$. The Kazhdan-Lusztig polynomial is difficult to construct explicitly and we will not give details of the construction in here. A recursive formula for the computation of the Kazhdan-Lusztig polynomial can be found in the original paper [20, Equation 2.2c].

We will show by induction, on $l(w)$, that T_w can be expressed as a linear combination of elements of $\{C_x \mid x \in S_n\}$. We have that $C_1 = T_1$ and $C_{s_i} = T_{s_i} - qT_1$, where 1 stands for the trivial element of S_n , and s_i is the transposition $(i, i+1)$ in S . For an arbitrary $w \in S_n$ we have

$$C_w = \left(\sum_{y < w} (-q)^{l(w)-l(y)} \overline{P_{y,w}(q^2)} T_y \right) + T_w.$$

Since $l(y) < l(w)$, by the inductive hypothesis we have that T_y can be expressed as a linear combination of C_x , for elements $x \in S_n$ with $l(x) < l(w)$. Furthermore, since the cardinality of $\{C_x \mid x \in S_n\}$ is equal to the cardinality of $\{T_x \mid x \in S_n\}$, then the set $\{C_x \mid x \in S_n\}$ forms a basis for $H(q, n)$.

Construction of W -graphs and cells. Consider $y, w \in S_n$. We say that $y \prec w$ if $y < w$, if $(-1)^{l(y)} = -(-1)^{l(w)}$, and if the Kazhdan-Lusztig polynomial $P_{y,w}$ has degree exactly $\frac{1}{2}(l(w) - l(y) - 1)$. We take $\mu(y, w)$ to be the coefficient of the highest power of q in $P_{y,w}$. Let Γ be the graph whose vertices X correspond to the $n!$ elements of S_n and whose edges are subsets of S_n of the form $\{y, w\}$ with $y \prec w$. We set $I_w = \{s_i \in S \mid s_i w < w\}$. We define a preorder relation \leq_Γ on the set of vertices of Γ as follows. Two vertices x, x'

satisfy $x \leq_\Gamma x'$ if there exist a sequence of vertices $x = x_0, x_1, \dots, x_n = x'$ such that for each i , $(1 \leq i \leq n)$, $\{x_{i-1}, x_i\}$ is an edge and $I_{x_{i-1}} \not\subseteq I_{x_i}$. Define the equivalence relation $x \sim_\Gamma x'$ if $x \leq_\Gamma x' \leq_\Gamma x$. The equivalence classes of S_n under \sim_Γ , denoted by $[w]$, are called *cells*. We denote by $\Gamma_{[w]}$ the subgraph of Γ whose vertices correspond to elements of $[w]$. Now we can define the action of T_{s_i} on $\{C_w \mid w \in S_n\}$:

$$T_{s_i}C_w = qC_w + C_{s_iw} + \sum_{y \prec w, s_iy < y} \mu(y, w)C_y.$$

This action above is well defined [21, Theorem 2.5].

Let D_w be the $\mathbb{Z}[q^{\pm 1}]$ -module spanned by the set $\{C_y \mid y \leq_\Gamma w\}$, and let D'_w be the $\mathbb{Z}[q^{\pm 1}]$ -module spanned by the set $\{C_y \mid y \leq_\Gamma w, y \notin [w]\}$. It is obvious that D'_w is contained in D_w . We will show that D_w and D'_w are left-ideals of $H(q, n)$. Recall the formula

$$T_{s_i}C_w = qC_w + C_{s_iw} + \sum_{y \prec w, s_iy < y} \mu(y, w)C_y.$$

The conditions $y \prec w, s_iy < y$ together with the existence of $\mu(y, w)$ show that $y \leq_\Gamma w$. Furthermore $s_iw \leq_\Gamma w$ since $w \prec s_iw$ (more particularly we have $P_{w, s_iw} = 1$ [20, Lemma 2.6 (iii)]), and $I_w \not\subseteq I_{s_iw}$. Hence we can define the quotient D_w/D'_w . We have the following theorem [21, Theorem 2.6]:

Theorem 3.2. *The graph $\Gamma_{[w]}$ defined above is a W-graph whose associated $\mathbb{Z}[q^{\pm 1}]$ -module is D_w/D'_w .*

By the theorem above we have a well defined representation

$$\rho : H(q, n) \rightarrow \text{End}(D_w/D'_w)$$

where the elements in the basis of D_w/D'_w correspond to the elements of the cell $[w]$. Kazhdan-Lusztig proved that the representation of $H(q, n)$ arising from the action on cells is irreducible and that the isomorphism class of the W-graph depends only on ρ and not on $[w]$ [20, Theorem 1.4].

Example of Figure 6. We have already seen that $s_2s_1 \leq_\Gamma s_1$. By denoting the trivial element of S_n by 1, we have that $1 \prec s_1$ with $s_1 \leq_\Gamma 1$. Then we have $1 \prec s_1s_2s_1$ with $1 \leq_\Gamma s_1s_2s_1$. Finally $s_1s_2s_1 \leq_\Gamma s_2s_1$. Hence $s_1 \sim_\Gamma s_2s_1$; the vertex set $\{s_1, s_2s_1\}$ is labeled by $\mu(s_1, s_2s_1) = 1$ [20, Theorem 2.6].

3.4 Robinson-Schensted correspondence and dual Knuth equivalence

In this section we will give an algorithm for finding elements of cells defined in the previous section. Unfortunately the construction of cells given by Kazhdan-Lusztig [20] is not easy when n is large. For that reason we will provide a different strategy to obtain cells. We have two tools: the Robinson-Schensted correspondence (RS-correspondence), and the dual Knuth equivalence. Using the RS-correspondence, we will be able to decide when two elements of S_n belong to the same cell. We will also be able to calculate the cardinality of a cell, that is, the dimension of the representation associated to that cell. By the dual Knuth equivalence, given a element $w \in S_n$ we will be able to obtain all elements of $[w]$. That is, by taking any $w \in S_n$, we will find all $y \in S_n$ such that $w \sim_\Gamma y$.

Robinson-Schensted correspondence. Here we will give an algorithm that associates a Young diagram to a given element of S_n . More precisely, in this algorithm every element of S_n corresponds to two standard tableaux of the same shape. For $w \in S_n$, let w_i denote the image of $w(i)$ under the mapping

$$w : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}.$$

We will identify w with the sequence $w_1 w_2 \dots w_n$. Consider an arbitrary tableaux T . We denote the boxes of the i^{th} row by $R_i(T)$. In the first step the row $R_1(T)$ contains a box filled by w_1 . In the next steps if w_j is greater than every number in $R_i(T)$ then we add a box on the right of all other boxes filled by w_j . If w_j is not greater than every number of $R_i(T)$, we consider $k \in R_i(T)$ such that k is the smallest number for which $w_j < k$. We replace k by w_j . If $R_{i+1}(T)$ does not exist, we add a box in $R_{i+1}(T)$ filled by k . If not, we repeat the same process with k playing the role of w_j in the $R_{i+1}(T)$ row. The algorithm ends when we insert all w_j in the boxes of T . The algorithm we described is called *row insertion algorithm*. The standard diagram we obtain is denoted by $P(w)$, and it is called *P-symbol*. We define the *Q-symbol* to be $Q(w) = P(w^{-1})$. For example in S_3 consider the element $s_1 s_2 = 231$, where $s_i \in S$ are transpositions. In Figure 7 we compute the P-symbol of $s_1 s_2$ step by step.

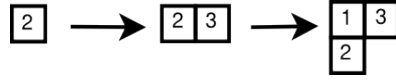


Figure 7: The P-symbol of $s_1 s_2$.

The pair $(P(w), Q(w))$ is called *Robinson-Schensted correspondence*. There is a bijection between the elements of S_n and the pairs $(P(w), Q(w))$ for all $w \in S_n$. For proof of Theorem 3.3 see [1, Theorem A].

Theorem 3.3. For $y, w \in S_n$ we have $y \sim_{\Gamma} w$ if and only if $Q(y) = Q(w)$.

As we have seen in the previous section Kashdan-Luzstig proved that the action of $H(q, n)$ on cells induces irreducible representations. But the connection of cells and Young diagrams is lacking. The latter connection is summarized in the following theorem [5, Theorems 6.5.2, 6.5.3]. For a fixed $x \in S_n$, we denote the shape of $Q(x)$ by $S(Q(x))$.

Theorem 3.4. The irreducible representation of $H(q, n)$ associated with the cell $[w]$, for a fixed $w \in S_n$, is labeled by a Young diagram λ , where $\lambda = S(Q(w))$.

Dual Knuth equivalence. Here we provide an algorithm to find all elements of a cell. That is, starting with an element of S_n , we have a process that will allow us to obtain elements of the cell associated to a given element of S_n [5, Appendix A3.6].

For $x, y \in S_n$, we write $x \sim_{dK} y$ (x is dual Knuth equivalent to y) if x and y differ by transposition of two values i and $i + 1$, and either $i - 1$ or $i + 2$ occurs in a position between those of i and $i + 1$. For example

$$215436 \sim_{dK} 315426 \sim_{dK} 415326 \sim_{dK} 425316$$

shows that $215436 \sim_{dK} 425316$. We have the next Theorem [5, Fact A3.6.2].

Theorem 3.5. For $x, y \in S_n$, we have that $Q(x) = Q(y)$ if and only if $x \sim_{dK} y$.

The algorithm for construction of a cell is divided into 2 steps.

1. Fix an element w of S_n . By Theorem 3.4 we obtain the cardinality of $[w]$.
2. By the dual Knuth equivalence, we obtain all elements of $[w]$. For any $x \in [w]$ we deduce $I_x = \{s_i \in S \mid s_i x < x\}$.

In the previous section we defined a class of $\mathbb{Z}[q^{\pm 1}]$ -modules and an action of the Hecke algebras on these modules. Since the construction of these modules by definition is difficult, we can use the algorithm described above to construct them.

3.5 Facts about the Hecke algebra representation

We fix an element $w \in S_n$. Let $H(q, n) \rightarrow \text{End}(V)$ be a Hecke algebra representation where V is the $\mathbb{Z}[q^{\pm 1}]$ -module spanned by C_x for all $x \in [w]$ as described in Section 3.2. Since $H(q, n)$ is a quotient of the group algebra of B_n , there is a well defined representation $\pi_\lambda : B_n \rightarrow \text{End}(V)$. Furthermore, λ is the shape of the standard diagram of $Q(w)$.

Representations when the Young diagram is rectangular. We now focus on representations of B_{2g+2} when $g \geq 2$ and λ is a rectangular diagram. For $g \geq 2$ we want to construct a representation $\pi_\lambda : B_{2g+2} \rightarrow \text{End}(V_\lambda)$ such that the matrices in the image of π_λ follow a pattern when the g increases. First we want to find cells that correspond to rectangular diagrams. There is a difficulty here. Consider for example $w \in S_6$ such that $[w]$ is a cell associated to a rectangular diagram. Furthermore w is an element of S_8 . But by the RS-correspondence we can see that $[w]$ in S_8 does not always correspond to a rectangular diagram.

We can solve this problem by making good choices for cells. We denote the generators of S_{2g+2} by the transpositions s_i , $i < 2g+2$, and we consider the element $s_1 s_3 s_5 \dots s_{2g+1} \in S_{2g+2}$. Recall from Section 3.3 that for $x \in S_n$, $Q(x)$ is the \mathbf{Q} -symbol. By the RS-correspondence it is easy to check that the shape of $Q(s_1 s_3 s_5 \dots s_{2g+1})$, denoted by λ_{2g+2} , is rectangle as indicated in Figure 8 for $g = 5$.

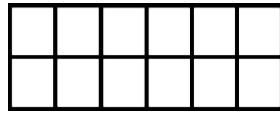


Figure 8: The Young diagram of the cell $[s_1 s_3 s_5 \dots s_{11}]$.

For $g \geq 2$, we consider the cells $[s_1 s_3 s_5] = W_1$, $[s_1 s_3 s_5 \dots s_{2g+1}] = W_2$. The cell $[s_1 s_3 s_5]$ contains the elements

$$\{s_1 s_3 s_5, s_1 s_4 s_3 s_5, s_2 s_1 s_3 s_5, s_2 s_1 s_4 s_3 s_5, s_3 s_2 s_1 s_4 s_3 s_5\}.$$

For $g = 3$ the cell $[s_1 s_3 s_5 s_7]$ contains the elements

$$\begin{aligned} &\{s_1 s_3 s_5 s_7, s_1 s_4 s_3 s_5 s_7, s_2 s_1 s_3 s_5 s_7, s_2 s_1 s_4 s_3 s_5 s_7, s_3 s_2 s_1 s_4 s_3 s_5 s_7, \\ &s_1 s_3 s_6 s_5 s_7, s_1 s_5 s_4 s_3 s_6 s_5 s_7, s_2 s_1 s_4 s_3 s_6 s_5 s_7, s_2 s_1 s_5 s_4 s_3 s_6 s_5 s_7, s_2 s_1 s_3 s_6 s_5 s_7 \} \end{aligned}$$

$$s_1 s_4 s_3 s_6 s_5 s_7, s_3 s_2 s_1 s_4 s_3 s_6 s_5 s_7, s_3 s_2 s_1 s_5 s_4 s_3 s_6 s_5 s_7, s_4 s_3 s_2 s_1 s_5 s_4 s_3 s_6 s_5 s_7\}.$$

The first five elements of $[s_1 s_3 s_5 s_7]$ above differ from the elements of $[s_1 s_3 s_5]$ by the generator s_7 . For $g > 3$ the first five elements of $[s_1 s_3 s_5 \dots s_{2g+1}]$ differ from $[s_1 s_3 s_5]$ by the word $s_7 s_9 \dots s_{2g+1}$.

Recall from Section 2 the correspondence $\sigma_i \mapsto T_{s_i}$ where T_{s_i} is a generator of $H(q, n)$. We have the following theorem.

Theorem 3.6. *The map $C_w \mapsto C_{ws_7 s_9 \dots s_{2g+1}}$ for $w \in W_1$, and $ws_7 s_9 \dots s_{2g+1} \in W_2$ defines an embedding for $H(q, 6)$ -modules $V_{\lambda_6} \hookrightarrow V_{\lambda_{2g+2}}|_{H(q, 6)}$.*

In other words we get a representation $B_{2g+2} \rightarrow \text{End}(V_{\lambda_{2g+2}})$, such that

$$\sigma_i \mapsto \pi_{\lambda_{2g+2}}(\sigma_i) = \begin{pmatrix} \pi_{\lambda_6}(\sigma_i) & A \\ 0 & B \end{pmatrix}$$

where $\pi_{\lambda_6}(\sigma_i)$ is a 5×5 matrix, d is the dimension of $\pi_{\lambda_{2g+2}}(\sigma_i)$, A is a matrix of dimension $5 \times (d - 5)$, and B is a matrix of dimension $(d - 5) \times (d - 5)$.

Proof. We denote the elements of W_1 by u_1, u_2, u_3, u_4, u_5 . Then the first five elements of W_2 have the form $u_i s_7 s_9 \dots s_{2g+1} = w_i$, where $i \leq 5$. Recall from Section 3.2 that the action of T_{s_j} on C_{w_i} is defined as follows:

$$T_{s_j} C_{w_i} = q C_{w_i} + C_{s_j w_i} + \sum_{y \prec w_i, s_j y < y} \mu(y, w_i) C_y.$$

We divide the proof into two steps. In the first step we show that the basis elements C_x , $x \in W_2$ in the above sum are the same for any $g \geq 2$. In the second step we show that $\mu(y', w_i) = \mu(y, u_i)$ for $y' \prec w_i, s_j y' < y'$, $y \prec u_i, s_j y < y$, and y, y' differ by the word $s_7 s_9 \dots s_{2g+1}$.

Step 1. By Theorem 3.2 if $s_j w_i < w_i$, then $T_{s_j} C_{w_i} = -q^{-1} C_{w_i}$; if $s_j w_i > w_i$ then the element $C_{s_j w_i}$ vanishes in $V_{\lambda_{2g+2}}$ (the $\mathbb{Z}[q^{\pm 1}]$ -module spanned by C_x for all $x \in W_2$).

We will show that if $y \prec w_i, s_j y < y$, then y is one of the first five elements in W_2 . By considering a weaker restriction $y \leq w_i$, we will show that either $y \notin W_2$ or y is one of w_i . We can see that the argument of the first step is true for $g = 2$, and 3 (see the examples of cells above). If $g \geq 4$ we note that u_i differ from w_i by the word $s_7 s_9 \dots s_{2g+1}$, and every u_i is a word in $\{s_j \mid j \leq 5\}$. Therefore, by the RS-correspondence for $y \in S_{2g+2}$ such that $y \leq w_i$ we have that either $y \notin W_2$ or y is one of w_i .

Step 2. If $k = 1$, and $i = 5$ do not occur simultaneously, then $\mu(w_k, w_i) = \mu(u_k, u_i) = 1$ since $l(w_i) - l(w_k) \leq 2$ [20, Theorem 2.6]. Where $l : S_{2g+2} \rightarrow \mathbb{Z}$ is the length function defined in Section 2. It remains to prove that $\mu(w_1, w_5) = \mu(u_1, u_5) = 1$. We will compute the polynomial P_{w_1, w_5} . We know that $\deg(P_{w_1, w_5}) = \frac{l(w_2) - l(w_1) - 1}{2} = 1$, hence $P_{w_1, w_2} = \mu(w_1, w_2)q + 1$. By the recursive formula given by Kazhdan-Lusztig [20, 2.2c] we have

$$P_{w_1, w_5} = P_{s_3 w_1, s_3 w_5} + q P_{w_1, s_3 w_5} - \sum \mu(z, s_3 w_5) q P_{w_1, z}$$

where the sum is over all z such that $w_1 \leq z \prec s_3 w_5, s_3 z < z$. But since we have $P_{w_1, s_3 w_5} = \mu(z, s_3 w_5) = P_{w_1, z} = 1$, [20, theorem 2.6] we conclude

$$P_{w_1, w_5} = P_{s_3 w_1, s_3 w_5} + q - \sum q.$$

Furthermore, by the RS-correspondence we can easily check that the elements s_3w_1 and s_3w_5 are not equivalent, which implies that the degree of $P_{s_3w_1, s_3w_5}$ is strictly less than 1. Hence $P_{s_3w_1, s_3w_5} = 1$. Finally, the kazhdan-lusztig polynomials have nonnegative integer coefficients [8, Corollary 1.2]. Therefore $P_{w_1, w_5} = q + 1$, and we have deduced that $\mu(w_1, w_5) = \mu(u_1, u_5) = 1$. \square

So far we have constructed irreducible representations of the braid groups B_n that factor through Hecke algebras $H(q, n)$. Theorem 3.6 is the key that we need to prove the Theorem A in Section 5.

4 Representation for the mapping class group a punctured sphere

For $i = 1, 2, \dots, 2g + 1$, let σ_i, H_i be the generators of $B_{2g+2}, \text{Mod}(\Sigma_{0, 2g+2})$ respectively as in Section 2. Since the homomorphism $B_{2g+2} \rightarrow \text{Mod}(\Sigma_{0, 2g+2})$ is surjective, it is reasonable to ask whether we can define a linear representation of $\text{Mod}(\Sigma_{0, 2g+2})$ via $H(q, 2g + 2)$. An affirmative answer was given by Jones [18, Theorem 10.2]. Since we use a different definition of $H(q, n)$ than Jones, we will reformulate his representation. But first we need to prove two lemmas.

We denote by f_i the image of σ_i in $\text{End}(V)$, and for $q^2 \neq -1$ we set $e_i = (q - f_i)/(q + q^{-1})$. In this Section we examine certain properties of the representation of the braid group $\pi_\lambda : B_n \rightarrow \text{End}(V)$. Firstly, we will calculate the image of the center of B_n under the map π_λ (Lemma 4.2).

Lemma 4.1. *We have $e_i^2 = e_i$. The rank of the idempotent e_i is the number of descending paths from diagram $\lambda_0 = \square$ to diagram λ of Young's lattice described in Section 3.1.*

Proof. We can easily check that $e_i^2 = e_i$. The rank of an idempotent in $\text{End}(V)$ is equal to its trace. Since all generators of B_n are conjugate in B_n , then all e_i have the same rank (trace). It suffice to calculate the rank of e_1 . We have that $\pi_{\lambda_0}(\sigma_1) = -q^{-1}$. Thus $\pi_{\lambda_0}(e_1) = 1$. We will prove the lemma by induction. We restrict the representation of B_n to B_{n-1} . Then the image of σ_1 in $\text{End}(V)$ is $\bigoplus \pi_{\lambda_i}(\sigma_1)$. By the induction argument the rank of $\pi_{\lambda_i}(\sigma_1)$ is the number of paths from λ_0 to λ_i . Since the rank of $\pi_\lambda(\sigma_1)$ is the sum of ranks of $\pi_{\lambda_i}(\sigma_1)$, we conclude that the rank is equal to the number of paths from the diagram λ_0 to the diagram λ . \square

Lemma 4.2. *If $\dim(\pi_\lambda) = d$ and $\text{rank}(e_i) = r$ then*

$$\pi_\lambda((\sigma_1\sigma_2\dots\sigma_{n-1})^n) = q^{n(n-1)\frac{d-2r}{d}} Id_{\pi_\lambda}.$$

Proof. The element $(\sigma_1\sigma_2\dots\sigma_{n-1})^n$ is in the center of B_n ; thus $\pi_\lambda((\sigma_1\sigma_2\dots\sigma_{n-1})^n)$ is a diagonal matrix whose entries are all equal. Since

$\det(\pi_\lambda(\sigma_i)) = \det(f_i) = \det(q - qe_i - q^{-1}e_i) = \det(q)\det(1 - e_i - q^{-2}e_i) = (-1)^r q^{d-2r}$, we have that $\det(\pi_\lambda((\sigma_1\sigma_2\dots\sigma_{n-1})^n)) = q^{n(n-1)(d-2r)}$. Hence

$$\pi_\lambda((\sigma_1\sigma_2\dots\sigma_{n-1})^n) = \omega q^{n(n-1)\frac{d-2r}{d}}$$

for some d^{th} root of unity ω . But ω depends continuously on q . But if we put $q = 1$, then we obtain a representation for the symmetric group S_n . But the center of S_n is trivial. Hence $\omega = Id_{\pi_\lambda}$. \square

Note. For a rectangular Young diagram λ_{2g+2} as in Theorem 3.6, the dimension of the associated representation is equal to

$$\binom{2g+2}{g+1} \frac{1}{g+2}.$$

Furthermore, according to Lemma 4.1 the rank of the idempotent e_i is equal to the descending paths from \square to λ_{2g+2} of the Young's lattice. But the latter is equal to the number of the descending paths from \square to λ_{2g} , which is equal to

$$\binom{2g}{g} \frac{1}{g+1}.$$

Now we will use Lemmas 4.1 and 4.2 to construct a representation for $\text{Mod}(\Sigma_{0,2g+2})$.

Proposition 4.3. *Consider the representation $\pi_\lambda : B_{2g+2} \rightarrow \text{End}(V_\lambda)$ associated to the Young diagram λ . We set $\pi'_\lambda(\sigma_i) = q^{(2r-d)/d} \pi_\lambda(\sigma_i)$. Then the map*

$$J : \text{Mod}(\Sigma_{0,2g+2}) \rightarrow \text{End}(V_\lambda)$$

defines a representation via $J(H_i) = \pi'_\lambda(\sigma_i)$ if and only if λ is rectangular.

We note that the homomorphism $J : \text{Mod}(\Sigma_{0,2g+2}) \rightarrow \text{End}(V_\lambda)$ is called the Jones representation.

$$\begin{array}{ccc} B_{2g+2} & \xrightarrow{\pi_\lambda} & \text{GL}_d(\mathbb{Z}[q^{\pm 1}]) \\ \downarrow & \nearrow \pi'_\lambda & \\ \text{Mod}(\Sigma_{0,2g+2}) & & \end{array}$$

Proof. We will show that the elements $\pi'_\lambda(\sigma_i)$ satisfy the relations of $\text{Mod}(\Sigma_{0,2g+2})$ defined in Section 2. First we assume that λ is rectangular. The braid relation and the disjointness relation are satisfied for $\pi'_\lambda(\sigma_i)$. By Lemma 4.2 we have that $\pi'_\lambda((\sigma_1 \sigma_2 \dots \sigma_{2g+1})^{2g+2})$ is trivial. We note that in the braid group we have the relation

$$\sigma_1 \sigma_2 \dots \sigma_{2g+1}^2 \sigma_{2g} \dots \sigma_1 = (\sigma_1 \sigma_2 \dots \sigma_{2g+1})^{2g+2} (\sigma_2 \dots \sigma_{2g+1})^{-(2g+1)}.$$

Then, the condition $\pi'_\lambda(\sigma_1 \sigma_2 \dots \sigma_{2g+1}^2 \sigma_{2g} \dots \sigma_1) = 1$ is equivalent to $\pi'_\lambda((\sigma_2 \dots \sigma_{2g+1})^{2g+1}) = 1$, since we already have that $\pi'_\lambda((\sigma_1 \sigma_2 \dots \sigma_{2g+1})^{2g+2})$ is trivial. But the restriction $\pi'_\lambda|_{B_{2g+1}}$ when λ is rectangular, satisfies the relation $\pi'_\lambda(\sigma_2 \dots \sigma_{2g+1})^{2g+1} = 1$. We note that by the Young's lattice, the dimension d and the rank r after the restriction above do not change. This proves the 'if' part.

If λ is not rectangular then π'_λ restricted to B_{2g+1} reduces as the direct sum of representations π'_{λ_i} . For $i \leq k$ and each π'_{λ_i} we have the numbers r_i and d_i such that $d = \sum_{i=1}^k d_i$ and $r = \sum_{i=1}^k r_i$. But the only way to have $\pi'_{\lambda_i}(\sigma_2 \dots \sigma_{2g+1})^{2g+1} = 1$ for $(d_i - 2r_i)/d_i$ is to be $(d - 2r)/d$ for all i . But this is impossible if $k > 1$. \square

Precomposing the map of Proposition 4.3 with the surjective homomorphism

$$\text{SMod}(\Sigma_g) \rightarrow \text{Mod}(\Sigma_{0,2g+2})$$

we obtain the following corollary.

Corollary 4.4. *There is a well defined representation $\text{SMod}(\Sigma_g) \rightarrow \text{End}(V_\lambda)$ defined by $T_{c_i} \mapsto q^{(2r-d)/d} \pi_\lambda(\sigma_i)$ if and only if λ is rectangular.*

By using the formula given in the definition of W-graphs, the action of $H(q, 6)$ on $[s_1 s_3 s_5]$ give the following matrices for $\text{Mod}(\Sigma_{0,2g+2})$

$$\begin{aligned}
H_1 &\mapsto q^{-1/5} \begin{pmatrix} -q^{-1} & 0 & 1 & 0 & 1 \\ 0 & -q^{-1} & 0 & 1 & 0 \\ 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 & q \end{pmatrix}, H_2 \mapsto q^{-1/5} \begin{pmatrix} q & 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 \\ 1 & 0 & -q^{-1} & 0 & 0 \\ 0 & 1 & 0 & -q^{-1} & 1 \\ 0 & 0 & 0 & 0 & q \end{pmatrix} \\
H_3 &\mapsto q^{-1/5} \begin{pmatrix} -q^{-1} & 1 & 1 & 0 & 0 \\ 0 & q & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 1 & -q^{-1} \end{pmatrix}, H_4 \mapsto q^{-1/5} \begin{pmatrix} q & 0 & 0 & 0 & 0 \\ 1 & -q^{-1} & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 \\ 0 & 0 & 1 & -q^{-1} & 1 \\ 0 & 0 & 0 & 0 & q \end{pmatrix} \\
H_5 &\mapsto q^{-1/5} \begin{pmatrix} -q^{-1} & 1 & 0 & 0 & 1 \\ 0 & q & 0 & 0 & 0 \\ 0 & 0 & -q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 & q \end{pmatrix}
\end{aligned}$$

In this section we constructed linear representations of the mapping class groups $\text{Mod}(\Sigma_{0,2g+2})$.

5 Proof of Theorem A

Representation over the normal closure of a power of a half-twist. Let $\mathcal{N}(h)$ denote the normal closure of h in $\text{Mod}(\Sigma_{0,2g+2})$. In this section we will use the Jones representation $J : \text{Mod}(\Sigma_{0,2g+2}) \rightarrow \text{End}(V_\lambda)$ to construct a linear representation for $\text{Mod}(\Sigma_{0,2g+2})/\mathcal{N}(H_i^n)$. As noted in the introduction, Humphries constructed a linear representation for the group $\text{Mod}(\Sigma_{0,6})/\mathcal{N}(H_i^n)$, and proved that $\text{Mod}(\Sigma_{0,2g+2})/\mathcal{N}(H_i^n)$ has infinite order if $n \geq 4$ [15, Theorem 4]. Our aim here is to extend Humphries' result for any $g \geq 2$.

Let λ be the Young diagram associated to the cell $[s_1 s_3 s_5 \dots s_{2g+1}]$ as before. Consider the representation $\pi_\lambda : B_{2g+2} \rightarrow \text{End}(V_\lambda)$. Recall from Section 4 that $\pi_\lambda(\sigma_i) = f_i$ and $e_i = (q - f_i)/(q + q^{-1})$ for $q^2 \neq -1$. Then we have $J(H_i) = q^{(2r-d)/d} (q - (q + q^{-1})e_i)$. We want to compute $J(H_i^n) = q^{n(2r-d)/d} (q - (q + q^{-1})e_i)^n$. Our aim is to modify the representation $J : \text{Mod}(\Sigma_{0,2g+2}) \rightarrow \text{End}(V_\lambda)$ such that the image of H_i^n will be trivial, and the relations of $\text{Mod}(\Sigma_{0,2g+2})$ will still hold. Then we will have a well defined linear representation for $\text{Mod}(\Sigma_{0,2g+2})/\mathcal{N}(H_i^n)$. By the binomial theorem, and the fact that $e_i^j = e_i$ for any $j \geq 2$ we have the following:

$$\begin{aligned}
(q - (q + q^{-1})e_i)^n &= \sum_{j=0}^n (-1)^j \binom{n}{j} q^{n-j} (q + q^{-1})^j e_i^j \\
&= q^n + \sum_{j=1}^n (-1)^j \binom{n}{j} q^{n-j} (q + q^{-1})^j e_i \\
&= q^n + e_i (-q^n + e_i \sum_{j=0}^n (-1)^j \binom{n}{j} q^{n-j} (q + q^{-1})^j) \\
&= q^n + e_i ((q - q - q^{-1})^n - q^n) \\
&= q^n + e_i ((-1)^n q^{-n} - q^n)
\end{aligned}$$

Hence we have

$$J(H_i^n) = q^{\frac{2nr}{d}} + e_i((-1)^n q^{2n\frac{r-d}{d}} - q^{\frac{2nr}{d}}). \quad (*)$$

Case n odd. It is convenient to change $q^{2/d}$ to t to obtain

$$J(H_i^n) = t^{nr} + e_i((-1)^n t^{n(r-d)} - t^{nr}).$$

We let $(-1)^{nt^d}$ be an n root of unity. Then we have $t^{nd} = -1$, and $t^n = (-1)^{1/d}$. If $J'(H_i) = (-1)^{-r/d} J(H_i)$, then $J'(H_i^n) = 1$. To see that the map

$$J' : \text{Mod}(\Sigma_{0,2g+2})/\mathcal{N}(H_i^n) \rightarrow \text{GL}_d(\mathbb{C}).$$

is a homomorphism, we only need to check that $J'(H_i)$ satisfy the relations of $\text{Mod}(\Sigma_{0,2g+2})$. In fact we only need to check that $(2g+1)(2g+2)r/d$ is even. In that case we would have $(-1)^{(2g+1)(2g+2)r/d} = 1$ (see proof of Proposition 4.3). In the end of Section 3, we gave two formulas for d, r in terms of g . A direct calculation shows that

$$(2g+1)(2g+2)\frac{r}{d} = (g+1)(g+2)$$

which is an even number.

Case n even. It is convenient to change $q^{1/d}$ to t to obtain

$$J(H_i^n) = t^{2nr} + e_i((-1)^n t^{2n(r-d)} - t^{2nr}).$$

We let t^d be n root of unity. Then $t^{(d-2r)n} J(T_{c_i}^n) = 1$. In this case we denote $t^{(d-2r)n} J(H_i)$ by $J'(H_i)$. We want to show that the map

$$J' : \text{Mod}(\Sigma_{0,2g+2})/\mathcal{N}(H_i^n) \rightarrow \text{GL}_d(\mathbb{C})$$

is a well defined representation. By Lemma 4.2 the number $(d-2r)(2g+1)(2g+2)$ is an integer multiple of d . Therefore $n(d-2r)(2g+1)(2g+2)$ is a multiple of nd . Since t^d is a root of unity, then $J'(H_i)$ satisfy the relations of $\text{Mod}(\Sigma_{0,2g+2})$ (see proof of Proposition 4.3).

We note here that the representations J' we constructed are not necessarily irreducible because the parameter t is a root of unity. But this will not be problem because our proof in the next section does not require irreducible representations for $\text{Mod}(\Sigma_{0,2g+2})/\langle H_i^n \rangle$.

Proof of Theorem A. Let $\Sigma_{0,2g+2}$ be a sphere with $2g+2$ marked points. We recall that $\text{Mod}(\Sigma_{0,2g+2})$ is generated by the half-twists H_i for $i \leq 2g+1$. We will prove that $A = (H_1 H_2)^6 H_3 (H_1 H_2)^6 H_3^{-1}$ has infinite order in $\text{Mod}(\Sigma_{0,2g+2})/\mathcal{N}(H_i^n)$. In fact we will prove that $J'(A)$ has infinite order in $\text{GL}_d(\mathbb{C})$, and $\text{GL}_d(\mathbb{C})$ respectively, by showing that if μ is an eigenvalue of either $J'(A)$, then μ^n will never be trivial for any $n \geq 4$; consequently this means that $J'(A)^n$ will never be the identity matrix for $n \geq 4$.

Case n even. By the construction of the representation we have

$$J'(A) = t^{\frac{12}{2}(2r-d)+(d-2r)n} \pi(\sigma_1\sigma_2)^6 \pi(\sigma_3) \pi(\sigma_1\sigma_2)^6 \pi(\sigma_3)^{-1}.$$

We denote by C the matrix $\pi(\sigma_1\sigma_2)^6 \pi(\sigma_3) \pi(\sigma_1\sigma_2)^6 \pi(\sigma_3)^{-1}$. We want to prove that C has infinite order. By Theorem 3.8 we have that the first 5×5 block of C are the same for any $g \geq 2$. Denote this 5×5 block by C' . We note that C' has 5 eigenvalues (2 distinct, and 1 repeated).

For $n = 6$, if $t^d = \exp(\pi i/3)$, then the absolute value of one of the eigenvalues of C' is approximately equals to 9.8989795. Hence it has infinite order. It follows that if n is even and it is divided by 3 then the eigenvalue has infinite order. For $n = 2(3k \pm 1) > 6$, put $t^d = \exp((4\pi i k)/n)$. If $k \rightarrow \infty$ then the absolute value of the same eigenvalue approximately converges to 9.8989795, thus it has infinite order. This completes the proof when n even.

Case n odd. By construction we have

$$J'(A) = (-1)^{\frac{-12r}{d}} t^{6(2r-d)} \pi(\sigma_1\sigma_2)^6 \pi(\sigma_3) \pi(\sigma_1\sigma_2)^6 \pi(\sigma_3)^{-1}.$$

We denote by C the matrix $\pi(\sigma_1\sigma_2)^6 \pi(\sigma_3) \pi(\sigma_1\sigma_2)^6 \pi(\sigma_3)^{-1}$. Again by Theorem 3.8 we have that the first 5×5 block of C are the same for any $g \geq 2$. We denote this 5×5 block by C' . The matrix C' has 3 eigenvalues (2 distinct and 1 repeated).

Let $(-1)^n t^d$ be an n^{th} root of unity. We set $t^d = \exp((3k\pi i)/n)$ where $n = (4k \pm 1)$ if k is odd, and $t^d = (-1)^{-n} \exp(((3k-1)\pi i)/n)$ where $n = (4k \pm 1)$ if k is even. In both cases if $k \rightarrow \infty$ then the absolute value of one of the eigenvalues of C' approximately converges rapidly to 9.5521659, thus it has infinite order. This completes the proof when n odd. \square

Now we can use Theorem A to prove Corollary 1.3.

Proof of Corollary 1.3. We want to prove that if $\mathcal{N}(\sigma_i^n)$ is the normal closure of σ_i^n in B_m , then $B_m/\mathcal{N}(\sigma_i^n)$ has infinite order when $n \geq 4$, and $m \geq 5$.

Recall the representations $J' : \text{Mod}(\Sigma_{0,2g+2})/\mathcal{N}(H_i^n) \rightarrow \text{GL}_d(\mathbb{C})$, when n is even, and $J' : \text{Mod}(\Sigma_g)/\mathcal{N}(H_i^n) \rightarrow \text{GL}_d(\mathbb{C})$ when n is odd. By the surjective homomorphism $B_{2g+2} \twoheadrightarrow \text{Mod}(\Sigma_{0,2g+2})$ defined by $\sigma_i \mapsto H_i$ we have

$$B_{2g+2}/\mathcal{N}(\sigma_i^n) \twoheadrightarrow \text{Mod}(\Sigma_{0,2g+2})/\mathcal{N}(H_i^n) \rightarrow \text{GL}_d(\mathbb{C})$$

when n is even, and

$$B_{2g+2}/\mathcal{N}(\sigma_i^n) \twoheadrightarrow \text{Mod}(\Sigma_{0,2g+2})/\mathcal{N}(H_i^n) \rightarrow \text{GL}_d(\mathbb{C})$$

Case m even. In this case the theorem follows by the surjectivity of

$$B_{2g+2}/\mathcal{N}(\sigma_i^n) \twoheadrightarrow \text{Mod}(\Sigma_{0,2g+2})/\mathcal{N}(H_i^n).$$

Case m odd. Here we want to prove that $B_{2g+1}/\mathcal{N}(\sigma_i^n)$ has infinite order. If we restrict the above representations to B_{2g+1} we get

$$B_{2g+1}/\mathcal{N}(\sigma_i^n) \rightarrow \text{GL}_d(\mathbb{C})$$

when n is even, and

$$B_{2g+1}/\mathcal{N}(\sigma_i^n) \rightarrow \mathrm{GL}_d(\mathbb{C})$$

when n is odd. These representations above are well defined, by the restriction formula described in Section 3.1. But by the proof of the Theorem A, the elements $(\sigma_1\sigma_2)^6\sigma_3(\sigma_1\sigma_2)^6(\sigma_3)^{-1}$, and $(\sigma_1\sigma_2)^6\sigma_3(\sigma_1\sigma_2)^6$ have infinite order in $B_{2g+1}/\mathcal{N}(\sigma_i^n)$.

To complete the proof, we denote by $\mathcal{N}'(\sigma_i^n)$ the normal closure of σ_i^n in B_4 , and we denote by $\mathcal{N}(\sigma_i^n)$ the normal closure of σ_i^n in B_5 .

We have that $B_4/(\mathcal{N}(\sigma_i^n) \cap B_4) < B_5/\mathcal{N}(\sigma_i^n)$, and $(\mathcal{N}(\sigma_i^n) \cap B_4)/\mathcal{N}'(\sigma_i^n) \triangleleft B_4/\mathcal{N}'(\sigma_i^n)$. By the third isomorphism theorem we get a surjective homomorphism

$$B_4/\mathcal{N}'(\sigma_i^n) \rightarrow B_4/(\mathcal{N}(\sigma_i^n) \cap B_4) < B_5/\mathcal{N}(\sigma_i^n).$$

But again the elements $(\sigma_1\sigma_2)^6\sigma_3(\sigma_1\sigma_2)^6$ and $(\sigma_1\sigma_2)^6\sigma_3(\sigma_1\sigma_2)^6(\sigma_3)^{-1}$ are in B_4 . Hence, $B_4/\mathcal{N}'(\sigma_i^n)$ has infinite order. \square

6 Proof of Theorem B

In this section we prove the Theorem B. Let $\mathcal{N}(H_i)$ denote the normal closure of a half-twist H_i in $\mathrm{Mod}(\Sigma_{0,2g+2})$. The representation $J : \mathrm{Mod}(\Sigma_{0,2g+2}) \rightarrow \mathrm{End}(V_\lambda)$ is defined by $J(H_i) = q^{(2r-d)/d}\pi_\lambda(\sigma_i)$, where $\sigma_i \in B_{2g+2}$, λ is a Young diagram as in Figure 8, and d is the dimension of the representation.

Proof of Theorem B Consider Equation (*) from previous section. For n even, let $q^{2/d}$ be a n^{th} root of unity. Then we have that $J(H_i^n)$ is trivial. By denoting J by J' we have a well defined linear representation

$$J' : \mathrm{Mod}(\Sigma_{0,2g+2})/\mathcal{N}(H_i^n) \rightarrow \mathrm{GL}_d(\mathbb{C}).$$

For n odd, let $-q^{2/d}$ be an n^{th} root of unity. In this case $(-1)^{-r/d}J(H_i^n)$ is trivial (see proof of Theorem A). We set $(-1)^{-r/d}J(H_i) = J'(H_i)$ and we have

$$J' : \mathrm{Mod}(\Sigma_{0,2g+2})/\mathcal{N}(H_i^n) \rightarrow \mathrm{GL}_d(\mathbb{C}).$$

Our aim is to prove that the group generated by the elements $J'(H_1^2), J'(H_2^2)$ contains a free nonabelian subgroup. By Young's lattice described in Section 3.1, if we restrict the Jones representation to the subgroup generated by H_1, H_2 then the representation J reduces into a direct sum of subrepresentations containing the representation labeled by $\lambda = \begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}$. Therefore, the elements $q^{(2r-d)/d}\pi_\lambda(\sigma_i) = \pi'_\lambda(\sigma_i)$ where $i = 1, 2$ are contained in the image of J . By Section 2.2 the map π_λ is the Burau representation. If q^2 is not a primitive root of unity of order in the set $\{1, 2, 3, 4, 6, 10\}$, then $\pi_\lambda(PB_3)$ contains a free nonabelian subgroups [12, Lemma 3.9].

Case n even. If $q^{2/d}$ be n^{th} nonprimitive root of unity such that $n \notin \{2, 4, 6, 10\}$, then $\pi'_\lambda(PB_3) < J'(G)$ contains a free nonabelian subgroups.

Case n odd. If we consider $-q^{2/d}$ be n^{th} nonprimitive root of unity such that $n \notin \{1, 3, 5\}$, then $\pi'_\lambda(PB_3) < J'(G)$ contains a free nonabelian subgroups.

References

- [1] Susumu Ariki. Robinson-Schensted correspondence and left cells. In *Combinatorial methods in representation theory (Kyoto, 1998)*, volume 28 of *Adv. Stud. Pure Math.*, pages 1–20. Kinokuniya, Tokyo, 2000.
- [2] J. S. Birman and T. E. Brendle. Braids: a survey. In *Handbook of knot theory*, pages 19–103. Elsevier B. V., Amsterdam, 2005.
- [3] Joan S. Birman. *Braids, links, and mapping class groups*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1974. *Annals of Mathematics Studies*, No. 82.
- [4] Joan S. Birman and Hugh M. Hilden. On the mapping class groups of closed surfaces as covering spaces. In *Advances in the theory of Riemann surfaces (Proc. Conf., Stony Brook, N.Y., 1969)*, pages 81–115. *Ann. of Math. Studies*, No. 66. Princeton Univ. Press, Princeton, N.J., 1971.
- [5] Anders Björner and Francesco Brenti. *Combinatorics of Coxeter groups*, volume 231 of *Graduate Texts in Mathematics*. Springer, New York, 2005.
- [6] H.S.M. Coxeter. Factor groups of the braid group. In *Proc. Fourth Canadian Math. Congress, Banff*, pages 95–122. 1957.
- [7] Persi Diaconis. *Chapter 7: Representation Theory of the Symmetric Group*, volume Volume 11 of *Lecture Notes-Monograph Series*, pages 131–140. Institute of Mathematical Statistics, Hayward, CA, 1988.
- [8] Ben Elias and Geordie Williamson. The Hodge theory of Soergel bimodules. *Ann. of Math. (2)*, 180(3):1089–1136, 2014.
- [9] B. Farb and D. Margalit. *A Primer on Mapping Class Groups (PMS-49)*. Princeton Mathematical Series. Princeton University Press, 2011.
- [10] J. S. Frame, G. de B. Robinson, and R. M. Thrall. The hook graphs of the symmetric groups. *Canadian J. Math.*, 6:316–324, 1954.
- [11] Louis Funar. On the TQFT representations of the mapping class groups. *Pac. J. Math*, 188, 1999.
- [12] Louis Funar and Toshitake Kohno. Free subgroups within the images of quantum representations. *Forum Math.*, 26(2):337–355, 2014.
- [13] M. Geck and G. Pfeiffer. *Characters of Finite Coxeter Groups and Iwahori-Hecke Algebras*. London Mathematical Society monographs. Clarendon Press, 2000.
- [14] Meinolf Geck and Nicolas Jacon. *Representations of Hecke algebras at roots of unity*, volume 15 of *Algebra and Applications*. Springer-Verlag London, Ltd., London, 2011.
- [15] Stephen P. Humphries. Normal closures of powers of Dehn twists in mapping class groups. *Glasgow Math. J.*, 34(3):313–317, 1992.
- [16] Stephen P. Humphries. Subgroups of pure braid groups generated by powers of dehn twists. *Rocky Mountain J. Math.*, 37(3):801–828, 06 2007.

- [17] StephenP. Humphries. Generators for the mapping class group. In Roger Fenn, editor, *Topology of Low-Dimensional Manifolds*, volume 722 of *Lecture Notes in Mathematics*, pages 44–47. Springer Berlin Heidelberg, 1979.
- [18] V. F. R. Jones. Hecke algebra representations of braid groups and link polynomials. *Ann. of Math. (2)*, 126(2):335–388, 1987.
- [19] Yasushi Kasahara. An expansion of the jones representation of genus 2 and the torelli group ii. *Journal of Knot Theory and Its Ramifications*, 13(02):297–306, 2004.
- [20] David Kazhdan and George Lusztig. Representations of Coxeter groups and Hecke algebras. *Invent. Math.*, 53(2):165–184, 1979.
- [21] Y. Yin. *W-graph representations for Coxeter groups and Hecke algebras*. PhD thesis, University of Sydney, 2004.